On the values of functions which in certain cases seem to be undetermined *

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§355 If any function y of x was a fraction $\frac{P}{Q}$ whose numerator and denominator vanish at the same time for a certain value of x, then in this case the fraction $\frac{P}{Q}$ expressing the value of the function y will become $= \frac{0}{0}$; since this expression can become equal to any either finite or infinite or infinitely small quantity, from it the value of y in this case cannot be concluded at all and hence seems undetermined. Nevertheless, it is easily seen, since except for in this case the function y always has a determined value, whatever is substituted for x, that also in this case the value of y cannot be undetermined. This will become obvious from the example $y = \frac{aa-xx}{a-x}$; having put $x = a$ it will be $y = \frac{0}{0}$, of course. But because having divided the numerator by the denominator it is $y = a + x$, it is evident, if one puts $x = a$, that it will be $y = 2a$ such that in this case that fraction $\frac{0}{0}$ becomes equal to the quantity $2a$.

§356 Therefore, since we showed above that $\frac{0}{0}$ can express any number, in examples of this kind a determined ratio which the numerator has to the denominator has to be investigated. But because in absolute zeros this diversity cannot be seen, instead of them infinitely small quantities have to be introduced; even if they in their nature do not differ from zero, nevertheless

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from the different functions of them which constitute the numerator and the denominator the value of the fraction becomes obvious immediately. So if one has this fraction $\frac{adx}{bdx}$, even if indeed the numerator and the denominator are $=0$, it is nevertheless plain that the value of this fractions is determined, namely $=\frac{a}{b}$. But if one has this fraction $\frac{adx^2}{bdx}$, this value will be zero, as this value of $\frac{adx}{bdx^2}$ is infinitely large. Therefore, if we introduce infinitely small quantities instead of the zeros which often enter the calculation, we will hence be able to find the ratio which the zeros have to each other soon and no further doubt about the meaning of expressions of this kind will remain.

§357 To render these things more clear, let us put that so the numerator as the denominator of this fraction $\frac{P}{Q}$ vanish, if one sets $x=a$. But to avoid these zeros which cannot be compared to each other let us put $x=a+dx$, which reduces to the first $x=a$ because of $dx=0$. But because, if one puts $x+dx$ instead of x, the functions P and Q go over into $p+dP$ and $Q+dQ$, this assumption $x=a+dx$ will be justified, if in these values one sets $x=a$ everywhere, in which case we assumed P and Q to vanish. Hence, if one puts $a+dx$ instead of x, the fraction $\frac{P}{Q}$ will be transformed into this one $\frac{dP}{dQ}$ which therefore expresses the value of the function $y=\frac{P}{Q}$ in the case $x=a$. And this expression cannot be undetermined any longer, if the true differentials of the functions P and Q are taken as we taught in the preceding chapter. For, this way the differentials dP and dQ never go over into absolutely zero, but, if they are not expressed by means of the differential dx itself, will at least be exhibited by means of its powers. Therefore, if one finds $dP=Rdx^m$ and $dQ=Sdx^n$, the value of the function $y=\frac{P}{Q}$ in the case $x=a$ will be $=\frac{Rdx^m}{Sdx^n}$ which will therefore be finite and $=\frac{R}{S}$, if it was $m=n$; but if it is $m>n$, then the value of the fraction will indeed be $=0$; but if $m<n$, this value grows to infinity.

§358 Therefore, if a fraction $\frac{P}{Q}$ of this kind occurs whose numerator and denominator in the case, say $x=a$, vanish at the same time the value of this fraction in this case $x=a$ will be found by means if the following rule:

Find the differentials of the quantities P and Q in the case $x=a$ and substitute them for P and Q having done which the fraction $\frac{dP}{dQ}$ will exhibit the value of the fraction $\frac{P}{Q}$ in question.

If the differentials dP and dQ found by means of the usual method become neither infinite nor vanish in the case $x = a$, then one can use them; but if both become either $= 0$ or $= \infty$, then these differentials must be investigated by means of the method explained in the preceding chapter in the case $x = a$. In most cases the calculation is miraculously contracted, if one puts $x - a = t$ or $x = a + t$ before, that a fraction $\frac{P}{Q}$ results whose numerator and denominator vanish in the case $t = 0$; for, then one will have the differentials dP and dQ, if one substitutes dt for t everywhere.

EXAMPLE 1

Let the value of this fraction $\frac{b-\sqrt{bb-tt}}{tt}$ *be in question in the case* $t = 0$.

Since in this case $t = 0$ both the numerator and the denominator vanish, only write dt instead of t and the value in question will be expressed by means of this fraction $\frac{b-\sqrt{bb-dt^2}}{dt^2}$. But because it is $\sqrt{bb-dt^2} = b - \frac{dt^2}{2b}$, this fraction goes over into this one $\frac{dt^2}{2bdt^2} = \frac{1}{2b}$. Therefore, the propounded fraction $\frac{b-\sqrt{bb-tt}}{tt}$ in the case $t = 0$ has the value $\frac{1}{2b}$.

EXAMPLE 2

Let the value of this fraction $\frac{\sqrt{aa+ax+xx}-\sqrt{aa-ax+xx}}{\sqrt{a+x}-\sqrt{a-x}}$ *be in question in the case* $x = 0$.

Here, one can again immediately substitute dx for x; because, having done this, it is

$$\sqrt{aa+adx+dx^2} = a + \frac{1}{2}dx + \frac{3dx^2}{8a},$$
$$\sqrt{aa-adx+dx^2} = a - \frac{1}{2}dx + \frac{3dx^2}{8a}$$

and

$$\sqrt{a+dx} = \sqrt{a} + \frac{dx}{2\sqrt{a}},$$
$$\sqrt{a-dx} = \sqrt{a} - \frac{dx}{2\sqrt{a}},$$

the numerator will become $= dx$ and the denominator $= \frac{dx}{\sqrt{a}}$, whence the value in question of the propounded fraction will be $= \sqrt{a}$.

EXAMPLE 3

Let the value of this fraction $\frac{x^3-4ax^2+7a^2x-2a^3-2a^2\sqrt{2ax-aa}}{xx-2ax-aa+2a\sqrt{2ax-xx}}$ *be in question in the case* $x = a$.

If the differentials are taken in usual manner and they are substituted for the numerator and denominator, one will have

$$\frac{3xx - 8ax + 7a^2 - 2a^3 : \sqrt{2ax - aa}}{2x - 2a + 2a(a - x) : \sqrt{2ax - xx}}$$

the numerator and denominator of which fraction vanish again, if one puts $x = a$. Therefore, for the same reason substitute their respective differentials again and this fraction will result

$$\frac{6x - 8a + 2a^4 : (2ax - aa)^{\frac{3}{2}}}{2 - 2a^3 : (2ax - xx)^{\frac{3}{2}}},$$

whose numerator and denominator again vanish in the case $x = a$. Therefore, we proceed to substitute their differentials for them

$$\frac{6 - 6a^5 : (2ax - aa)^{\frac{5}{2}}}{6a^3(a - x) : (2ax - xx)^{\frac{5}{2}}} = \frac{1 - a^5 : (2ax - aa)^{\frac{5}{2}}}{a^3(a - x) : (2ax - xx)^{\frac{5}{2}}}.$$

But also both the numerator and the denominator vanish again here for $x = a$. Therefore, further having substituted their differentials for them it will result

$$\frac{5a^6 : (2ax - aa)^{\frac{7}{2}}}{-(5a^5 - 8a^4x + 4a^3xx) : (2ax - xx)^{\frac{7}{2}}}.$$

Now, finally put a instead of x and this determined fraction will result $\frac{5:a}{-1:a^2} = -5a$ which is the value of the fraction in question.

But if, before this investigation is done, one puts $x = a + t$, the propounded fraction will be transformed into this one

$$\frac{2a^3 + 2a^2t - att + t^3 - 2a^2\sqrt{aa + 2at}}{-2aa + tt + 2a\sqrt{aa - tt}};$$

since this has the form $\frac{0}{0}$, if one puts $t = 0$, put dt instead of t and it will be

$$\frac{2a^3 + 2a^2dt - adt^2 + dt^3 - 2a^2\sqrt{aa + 2adt}}{-2aa + dt^2 + 2a\sqrt{aa - dt^2}}.$$

Now, convert the irrational formulas into series which must be continued until the rational terms are not longer cancelled:

$$\sqrt{aa + 2adt} = a + dt - \frac{dt^2}{2a} + \frac{dt^3}{2aa} - \frac{5dt^4}{8a^3},$$

$$\sqrt{aa - dt^2} = a - \frac{dt^2}{2a} - \frac{dt^4}{8a^3};$$

having substituted these values this fraction will result

$$\frac{5dt^4 : 4a}{-dt^4 : 4aa} = -5a$$

which is the value of the propounded fraction already found before.

EXAMPLE 4

To find the value of this fraction $\frac{a+\sqrt{2aa-2ax}-\sqrt{2ax-xx}}{a-x+\sqrt{aa-xx}}$ *in the case* $x = a$.

Having substituted their differentials for the numerator and the denominator this fraction will result which in the case $x = a$ will be equal to the propounded one:

$$\frac{-a : \sqrt{2aa - 2ax} - (a - x) : \sqrt{2ax - xx}}{-1 - x : \sqrt{aa - xx}}$$

whose numerator and denominator in the case $x = a$ become infinite. But if both are multiplied by $-\sqrt{a - x}$, one will have

$$\frac{a : \sqrt{2a} + (a - x)^{\frac{3}{2}} : \sqrt{2ax - xx}}{\sqrt{a - x} + x : \sqrt{a + x}}$$

which having put $x = a$ will give the determined value $\frac{a:\sqrt{2a}}{a:\sqrt{2a}} = 1$ which is therefore equal to the propounded fraction in the case $x = a$.

§359 Therefore, if one has a fraction $\frac{P}{Q}$ whose numerator and denominator vanish in the case $x = a$, one will be able to assign its value by means of the usual rules of differentiation and it will not be necessary to go back to the differentials which we treated in the preceding chapter. For, having taken the differentials the propounded fraction $\frac{P}{Q}$ in the case $x = a$ will become equal to the fraction $\frac{dP}{dQ}$; if its numerator and denominator take on finite values for $x = a$, one will find the value of the propounded fraction; but if the one becomes $= 0$ while the other remains finite, then the fraction will be either $= 0$ or $= \infty$, depending on whether the numerator or the denominator vanishes. But if one of the two or even both become $= \infty$, what happens, if one divides by quantities vanishing in the case $x = a$, then by multiplying both by these divisors the inconveniences is removed as it happened in the preceding example. But if so the numerator as the denominator vanish again, then just, as it was done in the beginning, the differentials are to be taken once more such that this fraction $\frac{ddP}{ddQ}$ results which in the case $x = a$ will still be equal to the propounded one; and if the same happens again in this fraction that it becomes $= \frac{0}{0}$, then substitute this one $\frac{d^3P}{d^3Q}$ for it and so forth, until one reaches a fraction which exhibits a determined value, either finite or infinitely large or infinitely small. So in the third example it was necessary to proceed to the fraction $\frac{d^4P}{d^4Q}$ before it was possible to assign the value of the propounded fraction $\frac{P}{Q}$.

§360 The use of this investigation becomes clear in the definition of the sums of series which we found above (chap. II § 22), if one outs $x = 1$. For, from the results that were derived there it follows that it will be:

$$
\begin{aligned}
x + x^2 \;\; + x^3 \;\; + \cdots + x^n &= \frac{x - x^{n+1}}{1 - x} \\
x + x^3 \;\; + x^5 \;\; + \cdots + x^{2n-1} &= \frac{x - x^{2n+1}}{1 - xx} \\
x + 2x^2 + 3x^3 + \cdots + nx^n &= \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} \\
x + 3x^3 + 5x^5 + \cdots + (2n-1)x^{2n-1} &= \frac{x + x^3 - (2n+1)x^{2n+1} + (2n-1)x^{2n+3}}{(1-xx)^2} \\
x + 4x^2 + 9x^3 + \cdots + n^2x^n &= \frac{x + x^2 - (n+1)x^{n+1} + (2nn + 2n - 1)x^{n+2} - nnx^{n+3}}{(1-x)^3}
\end{aligned}
$$

etc.

If now the sums of these series are desired in the case $x = 1$, in these expressions so the numerator as the denominator vanish. Therefore, the values of these sums in the case $x = 1$ can be defined by means of the method explained here. Since the same sums are known from elsewhere, from the agreement with those known values the validity of this method will become more clear.

EXAMPLE 1

To define the value of this fraction $\frac{x-x^{n+1}}{1-x}$ in the case $x = 1$ which will exhibit the sum of the series $1 + 1 + 1 + \cdots + 1$ consisting of n terms which therefore will be $= n$.

Since in the case $x = 1$ the numerator and the denominator vanish, substitute their differentials for them and one will have

$$\frac{1 - (n+1)x^n}{-1},$$

which for $x = 1$ gives n for the sum of the series in question.

EXAMPLE 2

To define the value of the fraction $\frac{x-x^{2n+1}}{1-xx}$ in the case $x = 1$ which will exhibit the sum of the series $1 + 1 + 1 + \cdots 1$ consisting of n terms which will therefore be $= n$.

Having taken the differentials the propounded fraction will be transformed into this one

$$\frac{1-(2n+1)x^{2n}}{-2x}$$

whose value having put $x=1$ will be $=n$.

EXAMPLE 3

To find the value of this fraction $\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$ *in the case* $x=1$ *which will express the sum of the series* $1+2+3+\cdots+n$ *which is known to be* $=\frac{nn+n}{2}$.

Having taken the differentials one gets to this fraction

$$\frac{1-(n+1)^2x^n+n(n+2)x^{n+1}}{-2(1-x)},$$

whose numerator and denominator still vanish in the case $x=1$. Therefore, take the differentials again so that this fraction results

$$\frac{-n(n+1)^2x^{n-1}+n(n+1)(n+2)x^n}{2},$$

which having put $x=1$ goes over into $\frac{n(n+1)}{2}=\frac{nn+n}{2}$ the sum of the propounded series.

EXAMPLE 4

To find the value of this fraction $\frac{x+x^3-(2n+1)x^{2n+1}+(2n-1)x^{2n+3}}{(1-xx)^2}$ *in the case* $x=1$ *which will express the sum of the series* $1+3+5+\cdots+(2n-1)$ *which is known to be* $=nn$.

Having substituted the differentials for the numerator and the denominator this fraction results

$$\frac{1+3xx-(2n+1)^2x^{2n}+(2n-1)(2n+3)x^{2n+2}}{-4x(1-xx)};$$

because this still has the same inconvenience that it goes over into $=\frac{0}{0}$ for $x=1$, take the differentials again

$$\frac{6x-2n(2n+1)^2x^{2n-1}+(2n-1)(2n+2)(2n+3)x^{2n+1}}{-4+12xx},$$

which for $x=1$ goes over into

$$\frac{6-2n(2n+1)^2+(2n-1)(2n+2)(2n+3)}{8}=nn.$$

EXAMPLE 5

To find the value of this fraction

$$\frac{x+x^2-(n+1)^2x^{n+1}+(2nn+2n-1)x^{n+2}-nnx^{n+3}}{(1-x)^3}$$

in the case $x=1$ *which will give the sum of the series* $1+4+9+\cdots+n^2$ *which is known to be* $=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$.

Having taken the differentials of the numerator and the denominator it will be

$$\frac{1+2x-(n+1)^3x^n+(n+2)(2nn+2n-1)x^{n+1}-nn(n+3)x^{n+2}}{-3(1-x)^2}$$

since in it the numerator and the denominator for $x=1$ vanish again, take the second differentials

$$\frac{2-n(n+1)^3x^{n-1}+(n+1)(n+2)(2nn+2n-1)x^n-n^2(n+2)(n+3)x^{n+1}}{6(x-1)}.$$

Since the same inconvenience is still present here, proceed to the third differentials that this fraction results

$$\frac{-n(n-1)(n+1)x^{n-2}+n(n+1)(n+2)(2nn+2n-1)x^{n-1}-n^2(n+1)(n+2)(n+3)x^n}{-6}$$

which for $x=1$ finally goes over into this determined form

$$\frac{-n(n-1)(n+1)^3+n(n+1)(n+2)(nn-n-1)}{-6}=\frac{n(n+1)(2n+1)}{6}=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n;$$

which is the value we found to express the mentioned series.

EXAMPLE 6

Let this fraction be propounded $\frac{x^m - x^{m+n}}{1 - x^{2p}}$ *whose value in the case* $x = 1$ *is to to be assigned.*

Since this fraction is the product of these two $\frac{x^m}{1+x^p} \cdot \frac{1-x^n}{1-x^p}$, but the value of the first factor in the case $x = 1$ is $= \frac{1}{2}$, it is only necessary that the value of the other factor $\frac{1-x^n}{1-x^p}$ in the same case is found which having taken the differentials will be $= \frac{nx^{n-1}}{px^{p-1}} = \frac{n}{p}$; therefore, the value of the propounded fraction in the case $x = 1$ will be $= \frac{n}{2p}$. The same value results, if the differentials are taken immediately in the propounded fraction; for, it will be

$$\frac{mx^{m-1} - (m+n)x^{m+n-1}}{-2px^{2p-1}},$$

whose value having put $x = 1$ will be $= \frac{-n}{-2p} = \frac{n}{2p}$ as before.

§361 The same method is to be used, if in the propounded fraction $\frac{P}{Q}$ either the numerator or the denominator or both were a transcendental number. That these operations are better explained, it seems advisable to add the following examples.

EXAMPLE 1

Let this fraction be propounded $\frac{a^n - x^n}{\ln a - \ln x}$ *whose value in the case* $x = a$ *is in question.*

Having taken the differentials one immediately gets to this form

$$\frac{-nx^{n-1}}{-1 : x} = nx^n$$

whose value for $x = a$ will be na^n.

EXAMPLE 2

Let this fraction be propounded $\frac{\ln x}{\sqrt{1-x}}$ *whose value in the case* $x = 1$ *is in question.*

Having taken the differentials of the numerator and the denominator this fraction results

$$\frac{1 : x}{-1 : 2\sqrt{1-x}} = \frac{-2\sqrt{1-x}}{x};$$

since its value for $x = 1$ is $= 0$, it follows that the fraction $\frac{\ln x}{\sqrt{1-x}}$ vanishes in the case $x = 1$.

EXAMPLE 3

Let this fraction be propounded $\frac{a-x-a\ln a-a\ln x}{a-\sqrt{2ax-xx}}$ *whose value for* $x = a$ *is in question in which case the numerator and the denominator vanish.*

Having differentiated the numerator and denominator according to the rule it will be

$$\frac{-1+a:x}{-(a-x):\sqrt{2ax-xx}} = \frac{(a-x)\sqrt{2ax-xx}}{-x(a-x)};$$

even if here the numerator and the denominator still vanish in the case $x = a$, nevertheless, since both are divisible by $a - x$, one will have this fraction $-\sqrt{\frac{2a-x}{x}}$ whose value in the case $x = a$ is determined and $= -1$; and therefore, the propounded fraction goes over into -1, if one puts $x = a$.

EXAMPLE 4

Let this fraction be propounded $\frac{e^x-e^{-x}}{\ln(1+x)}$ *whose value for* $x = 0$ *is in question.*

Having taken the differentials one will have this fraction

$$\frac{e^x + e^{-x}}{1:(1+x)}$$

which for $x = 0$ gives 2 for the value in question.

EXAMPLE 5

To find the value of this fraction $\frac{e^x-1-\ln(1+x)}{xx}$ *in the case in which one puts* $x = 0$.

If their differentials are substituted for the numerator and the denominator, this fraction will result

$$\frac{e^x - 1:(1+x)}{2x},$$

since which goes over into $\frac{0}{0}$, if one puts $x = 0$, take the differentials again that one has

$$\frac{e^x + 1 : (1+x)^2}{2}$$

which for $x = 0$ yields $\frac{1+1}{2} = 1$. The same is plain, if one immediately substitutes $0 + dx$ instead of x, because it is

$$e^{dx} = 1 + dx + \frac{1}{2dx^2} + \text{etc.} \quad \text{and} \quad \ln(1+dx) = dx - \frac{1}{2}dx^2 + \text{etc.},$$

$$\frac{e^{\ln x} - 1 - \ln(1+dx)}{dx^2} = \frac{dx^2}{dx^2} = 1.$$

EXAMPLE 6

Let the value of the fraction $\frac{x^n}{\ln x}$ in the case in which one puts $x = \infty$ be in question.

To reduce this fraction to a form which in this case goes over into $\frac{0}{0}$ represent it this way

$$\frac{1 : \ln x}{1 : x^n};$$

for, this in the case $x = \infty$ so the numerator as the denominator will vanish. Further, put $x = \frac{1}{y}$ such that in the case $x = \infty$ it is $y = 0$, and this fraction is propounded

$$-\frac{1 : \ln y}{y^n}$$

whose value in the case $y = 0$ must be investigated. But having taken the differentials it will be $\frac{1 : y(\ln y)^2}{ny^{n-1}} = \frac{1 : (\ln y)^2}{ny^n}$; since for $y = 0$ it goes over into $\frac{0}{0}$, take the differentials again and it will be $\frac{-2 : (\ln y)^3}{n^2 y^n}$; since here the same inconvenience is present, if the differentials are taken again, $\frac{6 : (\ln y)^4}{n^3 y^n}$ will result and so, no matter how far we proceed, always the same inconvenience occurs. Therefore, to find the value in question despite this obstacle, let s be the value of the fraction $-\frac{1 : \ln y}{y^n}$ in the case in which one puts $y = 0$, and because in the same case it also is

$$s = \frac{1 : (\ln y)^2}{ny^n},$$

from that equation it will be

$$ss = \frac{1 : (\ln y)^2}{y^{2n}}$$

which divided by the latter will give

$$s = \frac{ny^n}{y^{2n}} = \frac{n}{y^n},$$

from which it is understood that in the case $y = 0$ s becomes infinite. Therefore, the value of the fraction $-\frac{1:\ln y}{y^n}$ in the case $y = 0$ will be infinite and for $y = dx$ $\frac{1}{\ln dx}$ will have an infinite ratio to dx^n as we mentioned already above [§ 351].

EXAMPLE 7

Let the value of the fraction $\frac{x^n}{e^{-1:x}}$ be in question in the case $x = 0$ in which both the numerator and the denominator vanish.

Let $\frac{x^n}{e^{-\frac{1}{x}}} = s$ in this case; therefore, having taken the differentials it will also be

$$s = \frac{nx^{n-1}}{e^{-1:x} : xx} = \frac{nx^{n+1}}{e^{-1:x}},$$

and since here the same inconvenience occurs and always occurs again, no matter how far the differentiations are continued, we argue exactly as before. The first equation gives

$$x^n = e^{-1:x}s \quad \text{and} \quad x^{n(n+1)} = e^{-(n+1):x}s^{n+1};$$

the other equation gives

$$x^{n+1} = e^{-1:x}s : n,$$

whence it is

$$x^{n(n+1)} = e^{-n:x}s^n : n^n,$$

which value equated to the latter will give

$$e^{-1:x}sn^n = 1$$

and hence

$$s = \frac{1}{n^n e^{-1:x}} = \infty,$$

if $x = 0$. Therefore, having put x to be infinitely small dx^n will have an infinitely large ratio to $e^{-1:dx}$, no matter which finite number is substituted for n; therefore, it follows that $e^{-1:dx}$ is infinitely small and homogeneous to dx^m, if m was an infinitely large number.

EXAMPLE 8

Let the value of the fraction $\frac{1-\sin x+\cos x}{\sin x+\cos x-1}$ in the case in which one puts $x = \frac{\pi}{2}$ or equal to the arc of 90° be in question.

Having taken the differentials one will obtain this fraction

$$\frac{-\cos x - \sin x}{\cos x - \sin x}$$

which having put $x = \frac{\pi}{2}$ because of $\sin x = 1$ and $\cos x = 0$ goes over into 1 such that 1 is the value in question of the propounded fraction. The same is obvious without differentiation; for, since it is $\cos x = \sqrt{(1+\sin x)(1-\sin x)}$, the propounded fraction goes over into this one

$$\frac{\sqrt{1-\sin x} + \sqrt{1+\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}$$

which evidently becomes $= 1$, if $\sin x = 1$..

EXAMPLE 9

To find the value of this expression $\frac{x^x - x}{1-x+\ln x}$ in the case in which one puts $x = 1$.

Having substituted their differentials for the numerator and the denominator this fraction will result

$$\frac{x^x(1+\ln x) - 1}{-1 + 1 : x};$$

since this also becomes $= \frac{0}{0}$ for $x = 1$, take the differentials again that this fraction results

$$\frac{x^x(1+\ln x)^2 + x^x : x}{-1 : xx}$$

which having put $x = 1$ goes over into -2 which is the value of the propounded fraction in the case $x = 1$.

§362 Since here we decided to treat all expressions which in certain cases seem to have undetermined values, not only those fractions $\frac{P}{Q}$ whose numerator and denominator vanish in certain cases extend to this, but also fractions of such a kind whose numerator and denominator in a certain case become infinite are to be included, since their values also seem to be undetermined. If P and Q were functions of x of such a kind that in a certain case $x = a$ both become infinite and the fraction $\frac{P}{Q}$ takes on this form $\frac{\infty}{\infty}$, since two infinities as two zeros can have any ratio to each other, hence the value cannot be known at all. This case can be reduced to the preceding by transforming the fraction $\frac{P}{Q}$ into this form $\frac{1:Q}{1:P}$ the numerator and denominator of which fraction now vanish in the case $x = a$; and hence its value can be found by means of the method discussed before. But even without this transformation the value will be found, if not a but $a + dx$ is substituted for x; having done this not an absolute infinity ∞ will result, but it will be expressed as $\frac{1}{dx}$ or $\frac{A}{dx^n}$; even if these expressions are equally infinite as ∞, nevertheless having done the comparison of dx to its powers the value in question will easily be calculated.

§363 Also the products consisting of two factors of which the one in the certain case $x = a$ vanishes, but the other goes over into infinity extend to the same class; for, since every quantity can be represented by a product of this kind $0 \cdot \infty$, its value seems to be undetermined. Let PQ be a product of this kind in which, if one puts $x = a$, it is $P = 0$ and $Q = \infty$; its value will be found by means of the rules given before, if one puts $Q = \frac{1}{R}$; for, then the product PQ will be transformed into the fraction $\frac{P}{R}$ whose numerator and denominator both vanish in the case $x = a$; and hence its value can be investigated by means of the method explained before.

So, if the value of this product

$$(1 - x) \tan \frac{\pi x}{2}$$

in the case $x = 1$, in which it is $1 - x = 0$ and $\tan \frac{\pi x}{2} = \infty$, is in question, convert it into this fraction

$$\frac{1 - x}{\cot \frac{1}{2} \pi x}$$

whose numerator and denominator vanish in the case $x = 1$. Therefore, since the differential of the numerator is $1 - x = -dx$ and the differential of the denominator $\cot \frac{\pi x}{2} = -\frac{\pi dx:2}{(\sin \frac{1}{2}\pi x)^2}$, in the case $x = 1$ the value of the propounded fraction will be

$$= \frac{2}{\pi} \sin \frac{\pi x}{2} \cdot \sin \frac{\pi x}{2} = \frac{2}{\pi}$$

because of $\sin \frac{\pi}{2} = 1$.

§364 But especially those expressions which, if a certain value is attributed to x, go over into $\infty - \infty$ are included here; for, since two infinities can differ by any finite quantity, it is obvious that in this case the value of the expression is not determined, if the difference of those two infinities cannot be assigned. Therefore, this case occurs, if a function $P - Q$ of this kind is propounded in which for $x = a$ it is so $P = \infty$ as $Q = \infty$, in which case by means of the rules given before the value in question can not be assigned that easily. For, even if, having put that it this case it is $P - Q = f$, one sets $e^{P-Q} = e^f$ such that it is $e^f = \frac{e^{-Q}}{e^{-P}}$ where in the case $x = a$ so the numerator e^{-Q} as the numerator e^{-P} vanishes, if the rule given before is applied here, it will be $e^f = \frac{e^{-Q}dQ}{e^{-P}dP}$, whence because of $e^f = \frac{e^{-Q}}{e^{-P}}$ it would be $1 = \frac{dQ}{dP}$ and hence the value in question of f will not be found from this. If P and Q are algebraic quantities, since these only become infinite, if there are fractions whose denominators vanish, then $P - Q$ can be contracted into one single fraction whose denominator will vanish in the same way. If having done this the numerator vanishes, the value will be defined by means of the method explained above; but if the numerator does not vanish, then its value will indeed be infinite.

So if the value of this expression

$$\frac{1}{1-x} - \frac{2}{1-xx}$$

is desired in the case $x = 1$, since it goes over into

$$\frac{-1+x}{1-xx} = \frac{-1}{1+x},$$

it is plain that value in question is $= -\frac{1}{2}$.

§365 But if the functions P and Q were transcendental, then this transformation would lead to most cumbersome calculations in most cases. Therefore, it will be convenient that in these cases a direct method is used and instead of $x = a$, in which case both quantities P and Q go over into infinity, it is put $x = a + \omega$ while ω is an infinitely small quantity, for which one can take dx. If having done this it is $P = \frac{A}{\omega} + B$ and $Q = \frac{A}{\omega} + C$, it is obvious that the function $P - Q$ will go over into $B - C$ which value will be finite. Therefore, we will illustrate this method to investigate values of functions of this kind in the following examples.

EXAMPLE 1

Let the value of this expression $\frac{x}{x-1} - \frac{1}{\ln x}$ *be in question in the case in which one puts* $x = 1$.

Since so $\frac{x}{x-1}$ as $\frac{1}{\ln x}$ become infinite for $x = 1$, put $x = 1 + \omega$ and the propounded expression will be transformed into this one

$$\frac{1+\omega}{\omega} - \frac{1}{\ln(1+\omega)}.$$

Therefore, since it is

$$\ln(1+\omega) = \omega - \frac{1}{2}\omega^2 + \frac{1}{3}\omega^3 - \text{etc.} = \omega\left(1 - \frac{1}{2}\omega + \frac{1}{3}\omega^2 - \text{etc.}\right),$$

one will have

$$\frac{(1+\omega)(1 - \frac{1}{2}\omega + \frac{1}{3}\omega^2 - \text{etc.})}{\omega(1 - \frac{1}{2}\omega + \frac{1}{3}\omega^2 - \text{etc.})} = \frac{\frac{1}{2}\omega - \frac{1}{6}\omega^2 + \text{etc.}}{\omega\left(1 - \frac{1}{2}\omega + \frac{1}{3}\omega^2 - \text{etc.}\right)} = \frac{\frac{1}{2} - \frac{1}{6}\omega + \text{etc.}}{1 - \frac{1}{2}\omega + \frac{1}{3}\omega^2 - \text{etc.}}.$$

Now, having put ω to be infinitely small or $\omega = 0$ it is obvious that the value in question $= \frac{1}{2}$.

EXAMPLE 2

While e denotes the number whose hyperbolic logarithm is $= 1$ *and* π *the half of the circumference of the circle whose radius is* $= 1$, *to investigate the value of this expression* $\frac{\pi x - 1}{2xx} + \frac{\pi}{x(e^{2\pi x} - 1)}$ *in the case* $x = 0$.

This expression exhibits the sum of this series

$$\frac{1}{1+xx}+\frac{1}{4+xx}+\frac{1}{9+xx}+\frac{1}{16+xx}+\frac{1}{25+xx}+\text{etc.};$$

hence, if one puts $x=0$, the sum of this series must result

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.}$$

which is known to be $=\frac{\pi\pi}{6}$. But having put $x=0$ the value of the propounded expression

$$\frac{\pi x-1}{2xx}+\frac{\pi}{x(e^{2\pi x}-1)}$$

seems to be most undetermined because of all the infinite terms. Therefore, put $x=\omega$ while ω denotes an infinitely small quantity and the first term $\frac{\pi x-1}{2xx}$ goes over into

$$-\frac{1}{2\omega}^{2}+\frac{\pi}{2\omega}.$$

Further, since it is

$$e^{-2\pi\omega}-1=2\pi\omega+2\pi^2\omega^2+\frac{4}{3}\pi^3\omega^3+\text{etc.},$$

the other side $\frac{\pi}{x(e^{2\pi x}-1)}$ goes over into

$$\frac{\pi}{\omega(2\pi\omega+2\pi^2\omega^2+\frac{4}{3}\pi^3\omega^3)+\text{etc.}}=\frac{1}{2\omega^2(1+\pi\omega+\frac{2}{3}\pi^2\omega^2+\text{etc.})}.$$

But it is

$$\frac{1}{1+\pi\omega+\frac{2}{3}\pi^2\omega^2+\text{etc.}}=1-\pi\omega+\frac{1}{3}\pi^2\omega^2-\text{etc.},$$

whence the second term becomes

$$=\frac{1}{2\omega^2}-\frac{\pi}{2\omega}+\frac{1}{6}\pi^2-\text{etc.};$$

if to this the first is added, $\frac{1}{6}\pi^2$ results which is the value in question of the propounded expression in the case $x=0$.

The same can also be achieved by means of the method of fractions whose numerator and denominator vanish in a certain case; for, the propounded expression is transformed into this fraction

$$\frac{\pi x e^{2\pi x} - e^{2\pi x} + \pi x + 1}{2xxe^{2\pi x} - 2xx}$$

whose numerator and denominator vanish in the case $x = 0$. Therefore, having taken the differentials this fraction results

$$\frac{\pi e^{2\pi x} + 2\pi\pi x e^{2\pi x} - 2\pi e^{2\pi x} + \pi}{4xe^{2\pi x} + 4\pi xxe^{2\pi x} - 4x}$$

or this one

$$\frac{\pi - \pi e^{2\pi x} + 2\pi\pi e^{2\pi x}}{4xe^{2\pi x} + 4\pi xxe^{2\pi x} - 4x}$$

whose numerator and denominator still vanish, if one puts $x = 0$. Therefore, having taken the differentials again one will have

$$\frac{-2\pi\pi e^{2\pi x} + 2\pi\pi e^{2\pi x} + 4\pi^3 x e^{2\pi x}}{4e^{2\pi x} + 8\pi x e^{2\pi x} + 8\pi x e^{2\pi x} + 8\pi^2 xxe^{2\pi x} - 4}$$

or

$$\frac{\pi^3 x e^{2\pi x}}{e^{2\pi x} + 4\pi x e^{2\pi x} + 2\pi^2 x^2 e^{2\pi x} - 1}$$

or

$$\frac{\pi^3 x}{1 + 4\pi x + 2\pi^2 x^2 - e^{-2\pi x}}$$

whose numerator and denominator still vanish in the case $x = 0$. Therefore, take the differentials again

$$\frac{\pi^3}{4\pi + 4\pi^2 x + 2\pi e^{-2\pi x}}$$

which fraction for $x = 0$ goes over into $\frac{\pi^2}{6}$ as before.

EXAMPLE 3

While e and π retain the same values, let the value of this expression be in question in the case $x = 0$.

$$\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x}+1)}.$$

This expression is transformed into this one

$$\frac{\pi e^{\pi x} - \pi}{4xe^{\pi x} + 4x}$$

whose numerator and denominator vanish in the case $x = 0$. Therefore, put $x = \omega$, and because it is

$$e^{\pi\omega} = 1 + \pi\omega + \frac{1}{2}\pi^2\omega^2 + \frac{1}{6}\pi^3\omega^3 + \text{etc.},$$

the propounded formula is transformed into this one

$$\frac{\pi^2\omega + \frac{1}{2}\pi^3\omega^2 + \frac{1}{6}\pi^4\omega^3 + \text{etc.}}{8\omega + 4\pi\omega^2 + 2\pi^2\omega^3 + \text{etc.}}$$

which having put ω to be infinitely small immediately gives $\frac{1}{8}\pi^2$ which is the value in question of the propounded expression in the case $x = 0$. For, the propounded expression $\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x}+1)}$ will indeed exhibit the sum of this series

$$\frac{1}{1+xx} + \frac{1}{9+xx} + \frac{1}{25+xx} + \frac{1}{49+xx} + \text{etc.}$$

whose sum for $x = 0$ gives $= \frac{1}{8}\pi^2$.

EXAMPLE 4

To find the value of this expression $\frac{1}{2xx} - \frac{\pi}{2x\tan\pi x}$ in the case $x = 0$.

This propounded formula $\frac{1}{2xx} - \frac{\pi}{2x\tan\pi x}$ expresses the sum of this infinite series

$$\frac{1}{1-xx} + \frac{1}{4-xx} + \frac{1}{9-xx} + \frac{1}{16-xx} + \text{etc.}$$

If one therefore puts $x = 0$, the sum of the series

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.}$$

results which is $=\frac{1}{6}\pi\pi$. Since it is $\tan\pi=\frac{\sin\pi x}{\cos\pi x}$, the propounded expression will take on this form

$$\frac{1}{2xx}-\frac{\pi\cos\pi x}{2x\sin\pi x}=\frac{\sin\pi x-\pi x\cos\pi x}{2xx\sin\pi x}$$

whose numerator and denominator vanish in the case $x=0$. Therefore, put $x=\omega$, and because it is

$$\sin\pi\omega=\pi\omega-\frac{1}{6}\pi^3\omega^3+\text{etc.},\quad \cos\pi\omega=1-\frac{1}{2}\pi^2\omega^2+\text{etc.},$$

the propounded expression will be

$$\frac{\pi\omega-\frac{1}{6}\pi^3\pi^3+\text{etc.}-\pi\omega+\frac{1}{2}\pi^3\omega^3-\text{etc.}}{2\pi\omega^3-\frac{1}{3}\pi^3\omega^5+\text{etc.}}=\frac{\frac{1}{3}\pi^3\omega^3-\text{etc.}}{2\pi\omega^3-\text{etc.}}$$

which because of the infinitely small ω gives $\frac{1}{6}\pi^2$.

EXAMPLE 5

Since the sum of this infinite series is known, it is

$$\frac{1}{1-xx}+\frac{1}{9-xx}+\frac{1}{25-xx}+\frac{1}{49-xx}+\text{etc.}=\frac{\pi\sin\frac{1}{2}\pi x}{4x\cos\frac{1}{2}\pi x},$$

to find its sum, if it was $x=0$.

Since it is

$$\sin\frac{1}{2}\pi x=\frac{1}{2}\pi x-\frac{1}{48}\pi^3x^3+\text{etc.}\quad\text{and}\quad \cos\frac{1}{2}\pi x=1-\frac{1}{8}\pi^2x^2+\text{etc.},$$

the propounded expression will be

$$=\frac{\frac{1}{2}\pi^2x-\frac{1}{48}\pi^4x^3+\text{etc.}}{4x-\frac{1}{2}\pi^2x^3+\text{etc.}}=\frac{\frac{1}{2}\pi^2-\frac{1}{48}\pi^4x^2+\text{etc.}}{4-\frac{1}{2}\pi^2x^2+\text{etc.}};$$

if in this $x=0$, the value will obviously be $=\frac{1}{8}\pi^2$ which is the sum of the series

$$1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.}$$
as it was demonstrated above in several ways. But if one takes any even number for x, the sum of the propounded series is always $=0$.

§366 In these series we treated in the last two examples and containing the variable letter x one can attribute values of such a kind to x that certain terms grow to infinity in which cases the sum of the whole series will be infinite. So, one term of the series

$$\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\text{etc.},$$

if one substitutes any integer for x, because of the vanishing denominator always becomes infinite and therefore the series itself will become infinite. But if this infinite term is thrown out of the series, then the remaining sum will without any doubt be finite and will be expressed by means of the first series decreased by this infinite term in this way $\infty-\infty$; therefore, one will be able to find determined value, which will be seen more clearly from the examples added below.

EXAMPLE 1

To find the sum of the series

$$\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\text{etc.}$$

in the case $x=1$ and having subtracted the first term which in this case will be augmented to infinity.

Since the sum in general is

$$=\frac{1}{2xx}-\frac{\pi}{2x\tan\pi x},$$

the sum in question will be

$$=\frac{1}{2xx}-\frac{\pi}{2x\tan\pi x}-\frac{1}{1-xx}$$

for $x=1$. Let $x=1+\omega$ and for the sum in question one will have

$$\frac{1}{2(1+2\omega+\omega\omega)}-\frac{\pi}{2(1+\omega)\tan(\pi+\omega\pi)}+\frac{1}{2\omega+\omega\omega}.$$

But it is

$$\tan(\pi+\omega\pi)=\tan\omega\pi=\pi\omega+\frac{1}{3}\pi^3\omega^3+\text{etc.}$$

therefore, because the first term $\frac{1}{2xx}$ for $x=1$ has the determined value $\frac{1}{2}$, only the remaining terms are to be considered which will be

$$\frac{1}{\omega(2+\omega)}-\frac{\pi}{2\omega(1+\omega)(\pi+\frac{1}{3}\pi^3\omega^2)}=\frac{1}{\omega(2+\omega)}-\frac{1}{\omega(2+2\omega)(1+\frac{1}{3}\pi^2\omega^2)},$$

if ω is infinitely small in which case the term $\frac{1}{3}\pi^2\omega^2$ can be neglected. But it is

$$\frac{\omega}{\omega(2+\omega)(2+2\omega)}=\frac{1}{4}$$

for $\omega=0$ and therefore $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ is the sum of the series

$$\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\text{etc.}$$

as it is known from elsewhere.

EXAMPLE 2

To find the sum of the series

$$\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\text{etc.}$$

in the case in which one substitutes any integer n for x and omits the term $\frac{1}{nn-xx}$ of the series which would become infinite.

Therefore, this sum which is in question will be expressed this way

$$\frac{1}{2xx}-\frac{\pi}{2x\tan\pi x}-\frac{1}{nn-xx},$$

if one sets $x=n$, in which case the first term $\frac{1}{2xx}$ goes over into $\frac{1}{2nn}$, the two remaining ones on the other hand remain infinite. Therefore, put $x=n+\omega$, and because it is $\tan(\pi n + \pi\omega) = \tan \pi\omega = \pi\omega$ for infinitely small ω, for the sum in question we will have

$$\frac{1}{2nn} - \frac{\pi}{2(n+\omega)\pi\omega} + \frac{1}{2n\omega + \omega\omega}$$

or

$$\frac{1}{2nn} - \frac{1}{\omega(2n+2\omega)} + \frac{1}{\omega(2n+\omega)} = \frac{1}{2nn} + \frac{1}{(2n+2\omega)(2n+\omega)},$$

whence, if it was $\omega = 0$, the sum in question will result as

$$= \frac{1}{2nn} + \frac{1}{4nn} = \frac{3}{4nn}.$$

Therefore, it will be

$$\frac{3}{4nn} = \frac{1}{1-nn} + \frac{1}{4-nn} + \frac{1}{9-nn} + \cdots + \frac{1}{(n-1)^2 - nn}$$
$$+ \frac{1}{(n+1)^2 - nn} + \frac{1}{(n+2)^2 - nn} + \text{etc. to infinity}$$

or the sum of this series will be infinite

$$\frac{1}{(n+1)^2 - nn} + \frac{1}{(n+2)^2 - nn} + \frac{1}{(n+3)^2 - nn} + \text{etc.}$$
$$= \frac{3}{4nn} + \frac{1}{nn-1} + \frac{1}{nn-4} + \frac{1}{nn-9} + \cdots + \frac{1}{nn-(n-1)^2}.$$

EXAMPLE 3

To find the sum of this series

$$\frac{1}{1-xx} + \frac{1}{9-xx} + \frac{1}{25-xx} + \frac{1}{49-xx} + \text{etc.},$$

if one puts $x = 1$ and the first term $\frac{1}{1-xx}$ which in this case becomes infinite is subtracted.

Since the sum of this series in general is $= \frac{\pi \sin \frac{1}{2}\pi x}{4x \cos \frac{1}{2}\pi x}$, the sum in question will be

$$= \frac{\pi \sin \frac{1}{2}\pi x}{4x \cos \frac{1}{2}\pi x} - \frac{1}{1-xx},$$

if one puts $x = 1$. Since both of these terms become infinite, put $x = 1 - \omega$, and because it is

$$\sin\left(\frac{1}{2}\pi - \frac{1}{2}\pi\omega\right) = \cos\frac{1}{2}\pi\omega = 1 - \frac{1}{8}\pi^2\omega^2$$

and

$$\cos\left(\frac{1}{2}\pi - \frac{1}{2}\pi\omega\right) = \sin\frac{1}{2}\pi\omega = \frac{1}{2}\pi\omega$$

because of the infinitely small ω, one will have this expression

$$\frac{\pi(1-\frac{1}{8}\pi^2\omega^2)}{4(1-\omega)\frac{1}{2}\pi\omega} - \frac{1}{2\omega - \omega\omega} = \frac{1}{2\omega(2-2\omega)} - \frac{1}{\omega(2-\omega)}$$

which becomes $= \frac{1}{4}$ for $\omega = 0$, and therefore it is

$$\frac{1}{4} = \frac{1}{8} + \frac{1}{24} + \frac{1}{48} + \frac{1}{80} + \frac{1}{120} + \text{etc.}$$

Example 4

To find the sum of this series

$$\frac{1}{1-xx} + \frac{1}{9-xx} + \frac{1}{25-xx} + \frac{1}{49-xx} + \text{etc.},$$

if one substitutes any odd integer $2n-1$ *for* x *and this term* $\frac{1}{(2n-1)^2-xx}$ *which in this case becomes infinite is subtracted.*

Therefore, the sum which is in question will be

$$= \frac{\pi \sin \frac{1}{2}\pi x}{4x \cos \frac{1}{2}\pi x} - \frac{1}{(2n-1)^2 - xx}$$

for $x = 2n-1$. Therefore, let us set $x = 2n - 1 - \omega$ while ω is infinitely small and it will be

$$\sin\frac{1}{2}\pi x = \sin\left(\frac{2n-1}{2}\pi - \frac{1}{2}\pi\omega\right) = \pm\cos\frac{1}{2}\pi\omega,$$

where the upper sign holds, if n is an odd number, the lower, if it is even. In like manner it will be

$$\cos\frac{1}{2}\pi x = \cos\left(\frac{2n-1}{2}\pi - \frac{1}{2}\pi\omega\right) = \pm\sin\frac{1}{2}\pi\omega;$$

and hence, no matter whether n is even or odd, it will be

$$\frac{\sin\frac{1}{2}\pi x}{\cos\frac{1}{2}\pi x} = \frac{1}{\tan\frac{1}{2}\pi\omega} = \frac{1}{\frac{1}{2}\pi\omega}.$$

Therefore, the sum in question will be expressed this way

$$\frac{1}{2\omega(2n-1-\omega)} - \frac{1}{\omega(2(2n-1)-\omega)}$$

and therefore will be $= \frac{1}{4(2n-1)^2}$. So, if it $n = 2$, it will be

$$\frac{1}{36} = -\frac{1}{8} + \frac{1}{16} + \frac{1}{40} + \frac{1}{72} + \frac{1}{112} + \text{etc.}$$

the validity of which summation is known from elsewhere.